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FREE SETS FOR SOME SPECIAL SET MAPPINGS

BY

LUDOMIR NEWELSKI

Mathematical Institute of the Polish Academy of Sciences, Kopernika 18, 51–617 Wroclaw, Poland

ABSTRACT

In this paper we investigate under MA assumption the existence of free sets for $f: X \to \mathcal{P}(X)$ satisfying some special conditions. For example, we prove, assuming MA, that if $\kappa < 2^{\aleph_0}$ and $f: \mathbf{R} \to \mathrm{NWD}$, then there is a free set for f of cardinality κ.

1. Introduction

In this paper we use the standard set theoretical terminology. For any set A and any cardinal κ, we denote the sets $\{B \subseteq A : |B| < \kappa\}$, $\{B \subseteq A : |B| \leqq \kappa\}$ and $\{B \subseteq A : |B| = \kappa\}$ by $[A]^{<\kappa}$, $[A]^{\leqq\kappa}$ and $[A]^{\kappa}$ respectively. Here $\mathbf{R}$ denotes the set of all real numbers. We use the symbols $\mathbf{K}$ and NWD to denote the ideal of meager subsets of $\mathbf{R}$ and the family of all nowhere dense subsets of $\mathbf{R}$ respectively. $\mathcal{R}_\kappa$ denotes the measure algebra on 2^κ induced by the product measure; $\mathbf{K}^+ = \mathcal{P}(\mathbf{R}) \setminus \mathbf{K}$ and $\operatorname{cov} \mathbf{K} = \min\{|A| : A \subseteq \mathbf{K}\ \&\ \bigcup A = \mathbf{R}\}$. Here MA and CH denote Martin's axiom and Continuum hypothesis respectively.

Assume that $f: X \to \mathcal{P}(X)$. For $A \subseteq X$ we say that A is free for f if $x \notin f(y)$ for any $x \neq y \in A$. Throughout the paper we use the following abbreviation: if $f: X \to \mathcal{P}(X)$ and $S \subseteq X$ then by $f(S)$ we mean $\bigcup\{f(x): x \in S\}$. Moreover, we always, without loss of generality, assume that $x \in f(x)$. We define also $f^*: [X]^{<\aleph_0} \to \mathcal{P}([X]^{<\aleph_0})$ by $A \in f^*(B)$ iff $A \cap f(B) \neq \emptyset$.

In the paper we prove results concerning the existence of free sets for some special set mappings, such as mentioned for example in problems 36 and 38A from [4]. Problem 36 concerns the existence of free sets of arbitrarily large countable order type for $f: \omega_1 \to [\omega_1]^{\leqq\aleph_0}$ such that $f(x) \cap f(y)$ is finite for $x \neq y \in \omega_1$. 38A reads: Does there exist an uncountable free set for any

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f: $\mathbf{R} \to$ NWD? These problems are already solved. Namely, problem 36 is answered positively in ZFC (see [3], [10]). In Corollary 2.4 we strengthen (under MA assumption) this result. The answer to problem 38A is more complicated. Assuming CH, S. H. Hechler in [7] proved that there is f: $\mathbf{R} \to$ NWD without an uncountable free set. He proved also in [8] that it is consistent with ZFC+ $\neg$CH, that for any f: $\mathbf{R} \to$ NWD there is an uncountable free set. U. Avraham built in [1] a consistent with ZFC + $\neg$CH example of f: $\mathbf{R} \to$ NWD without an uncountable free set. In Lemma 4.1 we prove it in another way.

In Section 2 we formulate combinatorial conditions W1 and W2 implying the existence of free sets. In [6] it is proved that W2, assuming MA + $\neg$CH, implies the existence of uncountable free sets ([6, Lemma 42B]). We strengthen this by showing under the same assumptions the existence of dense uncountable free sets. An investigation of existence of free sets for f: $X \to \mathcal{P}(X)$ under MA assumption is carried out in §§41, 42 from [6]. However, the results proven there concern the existence of free sets of power $\leqq \aleph_1$ for set mappings with, essentially, countable values (compare theorem 41H, lemma 42B from [6]). In Section 3 we prove the main result of this paper, asserting (under MA assumption) the existence of a dense free set of arbitrary $< 2^{\aleph_0}$ power for f: $\mathbf{R} \to$ NWD, thus answering problem 2 from [8]. We also show in Lemma 3.5 that it is possible to prove the existence of free sets of power $> \aleph_1$ for some other f's with (possibly) uncountable values.

In Section 4 we complete an answer to problems from [8] and formulate some questions.

2. Two combinatorial theorems

In this section we investigate the existence of free sets for f: $\omega_1 \to [\omega_1]^{\leqq \aleph_0}$ satisfying some special conditions. These are:

W1. $\neg \exists \langle x_\alpha, y_\alpha : \alpha < \omega_1 \rangle$ s.t. $x_\alpha, y_\alpha \in \omega_1$ & $\forall \alpha, \beta < \omega_1$ $x_\alpha \in f(y_\beta) \leftrightarrow \alpha \leqq \beta$.

W2. $\neg \exists \langle \bar{x}_\alpha, \bar{y}_\alpha : \alpha < \omega_1 \rangle$ s.t. $\bar{x}_\alpha, \bar{y}_\alpha \in [\omega_1]^{<\aleph_0}$, $\forall \alpha < \beta < \omega_1$
$\bar{x}_\alpha \cap \bar{x}_\beta = \bar{y}_\alpha \cap \bar{y}_\beta = \varnothing$ & $\forall \alpha, \beta < \omega_1$ $\bar{x}_\alpha \in f^*(\bar{y}_\beta) \leftrightarrow \alpha \leqq \beta$.

W2 is a reformulation of the special case of the condition formulated in lemma 42B in [6] (see also [11]). Notice that W1 and W2 are similar to "strong cut condition" defined in [5] and that W2 implies W1.

Fact 2.1. Let f: $\omega_1 \to [\omega_1]^{\leqq \aleph_0}$. W1 is equivalent to (1) and W2 is equivalent to (2), where

(1) For any sequence $\langle y_\alpha : \alpha < \omega_1 \rangle \subseteq \omega_1$ there is $\alpha < \omega_1$ s.t. for every $\alpha < \beta < \omega_1$ we have $\bigcap\{f(y_\gamma): \alpha < \gamma < \omega_1\} = \bigcap\{f(y_\gamma): \beta < \gamma < \omega_1\}$.

(2) For any sequence $\langle \bar{y}_\alpha : \alpha < \omega_1 \rangle \subseteq [\omega_1]^{<\aleph_0}$ s.t. $\bar{y}_\alpha \cap \bar{y}_\beta = \emptyset$ for $\alpha \neq \beta$, there is $\alpha < \omega_1$ s.t. for every $\alpha < \beta < \omega_1$ we have

$$\bigcap\{f^*(\bar{y}_\gamma): \alpha < \gamma < \omega_1\} = \bigcap\{f^*(\bar{y}_\gamma): \beta < \gamma < \omega_1\}.$$

Proof of this fact is easy and we omit it. □

THEOREM 2.2. *Let $f: \omega_1 \to [\omega_1]^{\leqq \aleph_0}$ satisfy* W1 *and assume that $A_n \in [\omega_1]^{\aleph_1}$ for $n < \omega$. Then there is $A \subseteq \omega_1$ free for f and such that $A \cap A_n \neq \emptyset$ for all n.*

PROOF. We construct by induction on $k < \omega$ ordinals a_k and sets A_n^k for $n \geqq k$ satisfying the following conditions:

(1) $A_n^0 = A_n$ for $n < \omega$,

(2) $A_n^{k+1} \subseteq A_n^k$ and A_n^k is uncountable for every n, k,

(3) $a_k \in A_k^k$,

(4) for every $n > k$ and $a \in A_n^k$, $\{a_0, \dots, a_k, a\}$ is free for f.

Clearly, if we carry out this construction, the theorem will be proved with $A = \{a_n : n < \omega\}$. So assume that we have defined $a_0, \dots, a_{k-1}$ and A_n^i for $i \leqq k$ and $n \geqq i$. We have to define a_k and A_n^{k+1} for $n > k$. Let us consider the set

$$B = A_k^k \cap \bigcup_{n>k} \bigcup_{\alpha<\omega_1} \bigcap\{f(a): a \in A_n^k \setminus \alpha\}.$$

From Fact 2.1 we infer that this set is countable. So we can choose a_k to be any element of $A_k^k \setminus B$. For $n > k$ we define

$$A_n^{k+1} = \{a \in A_n^k : \{a, a_k\} \text{ is free for } f\}.$$

It suffices to show that A_n^{k+1} is uncountable for any $n > k$. Hence suppose that this is not the case for some $n > k$. Then there is $\alpha < \omega_1$ s.t. $f(a_k) \cup A_n^{k+1} \subseteq \alpha$, hence $a_k \in \bigcap\{f(a): a \in A_n^k \setminus \alpha\}$. This gives a contradiction. □

THEOREM 2.3 (MA + ¬CH). *Let $f: \omega_1 \to [\omega_1]^{\leqq \aleph_0}$ satisfy* W2 *and assume that $A_n \in [\omega_1]^{\aleph_1}$ for $n < \omega$. Then there is $A \in [\omega_1]^{\aleph_1}$ free for f and such that $|A \cap A_n| = \aleph_1$ for all n.*

PROOF. Let us consider the following notion of forcing:

$$P = \{B \in [\omega_1]^{<\aleph_0}: \forall n\, |\{a \in A_n : B \cup \{a\} \text{ is free for } f\}| = \aleph_1\}$$

ordered by reverse inclusion.

CLAIM 1. P satisfies ccc.

PROOF OF CLAIM 1. Suppose that $\{B_\alpha : \alpha < \omega_1\} \subseteq P$. By the Δ-system lemma we can assume w.l.o.g. that for some B we have $B_\alpha \cap B_\beta = B$ for any $\alpha \neq \beta < \omega_1$. Let us define $f_1 : \omega_1 \to [\omega_1]^{\leqq \aleph_0}$ by

$$\alpha \in f_1(\beta) \qquad \text{iff } (B_\alpha \setminus B) \in f^*(B_\beta).$$

W2 for f implies W1 for f_1, so by Fact 2.1 we find $\alpha < \omega_1$ such that

$$C_0 = \{\beta < \omega_1 : \alpha \neq \beta \text{ and } \{\alpha, \beta\} \text{ is free for } f_1\}$$

is uncountable. Let

$$C_{n+1} = \{a \in A_n : B_\alpha \cup \{a\} \text{ is free for } f\} \qquad \text{for } n < \omega.$$

W2 implies that there is a tail T of C_0 such that for all $\beta \in T$, $\{a \in C_{n+1} : B_\beta \cup \{a\}$ is free for $f\}$ is uncountable (as in the proof of Theorem 2.2). So for $\beta \in T$, $B_\alpha \cup B_\beta$ is free for f and for every n, $|\{a \in A_n : B_\alpha \cup B_\beta \cup \{a\} \text{ is free for } f\}| = \aleph_1$, hence $B_\alpha \cup B_\beta \in P$. □

CLAIM 2. For every $\alpha < \omega_1$ and $n < \omega$ the set $\{B \in P : B \cap (A_n \setminus \alpha) \neq \emptyset\}$ is dense.

PROOF of this claim is similar to that of Claim 1 and of Theorem 2.2. □

Theorem 2.3 now follows easily. □

REMARK. In [6], lemma 42B the existence of an uncountable free set is proved under the assumptions of 2.3. The notion of forcing used there is (essentially) $[\omega_1]^{<\aleph_0}$ ordered by

$$B_1 \geqq B_2 \qquad \text{iff } B_1 \subseteq B_2 \;\&\; (B_2 \setminus B_1) \in f^*(B_1).$$

Now it's time for some examples.

(1) Assume that $\gamma < \omega_1$ and $f : \omega_1 \to [\omega_1]^{\leqq \aleph_0}$ is such that $\mathrm{ot}(f(\alpha) \cap f(\beta)) < \gamma$ for $\alpha \neq \beta < \omega_1$. Then f satisfies W1 and W2, so in virtue of Theorem 2.3 we get

COROLLARY 2.4 (MA + ¬CH). *Assume that $\gamma < \omega_1$, $f : \omega_1 \to [\omega_1]^{\leqq \aleph_0}$ and for $\alpha \neq \beta < \omega_1$ $\mathrm{ot}(f(\alpha) \cap f(\beta)) < \gamma$. Let $A_n \in [\omega_1]^{\aleph_1}$ for $n < \omega$. Then there is $A \subseteq \omega_1$ free for f and such that $|A \cap A_n| = \aleph_1$ for every n.* □

Similarly, if $f : \omega_1 \to [\omega_1]^{\leqq \aleph_0}$ satisfies

$$(*) \qquad |\textstyle\bigcap\{f(a) : a \in C\}| < \aleph_0 \qquad \text{for any } C \in [\omega_1]^{\aleph_1},$$

then f satisfies W1. Notice that if MA + ¬CH holds, then (*) implies W2 as well (it is proved explicitly in the proof of lemma 42H from [6]). The author doesn't know if it is possible to construct f satisfying W1 that fails to satisfy W2 in ZFC only. However if we add some extra axioms, then such an f exists.

FACT 2.5. (1) (CH) There is $f\colon \omega_1 \to [\omega_1]^{\leqq \aleph_0}$ satisfying (*) that fails to satisfy W2.

(2) Let M be a transitive model of ZFC, and N be obtained from M by adding ω_1 Cohen generic reals. Then in N there is $f\colon \omega_1 \to [\omega_1]^{<\aleph_1}$ satisfying (*) that doesn't satisfy W2.

PROOF. Let $A = \{\alpha_0\colon \alpha < \omega_1\} \cup \{\alpha_1\colon \alpha < \omega_1\}$. We construct $f\colon A \to [A]^{\leqq \aleph_0}$ satisfying (*) that fails to satisfy W2. The failure to satisfy W2 is witnessed by $\bar{x}_\alpha = \bar{y}_\alpha = \{\alpha_0, \alpha_1\}$ for $\alpha < \omega_1$, and f satisfies

$$f(\alpha_0) \cup f(\alpha_1) = \{\beta_i\colon \beta < \alpha \text{ and } i = 0, 1\}.$$

so the only problem left is to ensure (*).

(1) (CH) Let $[A]^{\aleph_0} = \{C_\alpha\colon \omega \leqq \alpha < \omega_1\}$ and $C_\alpha \subseteq \{\beta_i\colon \beta \leqq \alpha,\ i = 0, 1\}$. Define $f(\alpha_i)$ $(i = 0, 1)$ for $\alpha < \omega$ arbitrarily and for $\alpha \geqq \omega$ in such a way that $f(\alpha_i)$ doesn't contain any C_β, for $\beta < \alpha$. Clearly such an f satisfies (*).

(2) Let $N = M[\langle c_\alpha\colon \alpha < \omega_1\rangle]$, where $\langle c_\alpha\colon \alpha < \omega_1\rangle$ is a sequence of Cohen generic reals, $c_\alpha \in {}^\omega 2$. As in (1) it suffices to define $f(\alpha_i)$ for infinite α's. For $\alpha \geqq \omega$ let $\alpha + 1 = \{\beta^n\colon n < \omega\}$. We define

$$\beta_j^n \in f(\alpha_j) \qquad \text{iff } c_\alpha(n) = 0.$$

Now, if C is an infinite, countable subset of A in N, then there is an α such that for $\beta > \alpha$, $f(\beta_i)$ doesn't contain C. □

(2) Assume that $f\colon \mathbf{R} \to \mathrm{NWD}$. W.l.o.g. $f(x)$ is closed and $x \in f(x)$ for all x. We shall frequently use the following

LEMMA 2.6. *Let κ be an uncountable regular cardinal. Then there is no collection $\{\langle s_\alpha, t_\alpha\rangle\colon \alpha < \kappa\}$ of pairs of nowhere dense sets of reals such that*

(a) s_α is bounded,

(b) $\mathrm{cl}(s_\alpha) \cap \mathrm{cl}(t_\beta) \neq \emptyset$ iff $\alpha \leqq \beta$.

PROOF. For $\alpha < \kappa$ choose an open set U_α (a finite sum of rational intervals) such that $\mathrm{cl}(s_{\alpha+1}) \subseteq U_\alpha$ and U_α is disjoint to $\mathrm{cl}(t_\alpha)$. For $\alpha \neq \beta$, $U_\alpha \neq U_\beta$, and there are only countably many possible U_α's, a contradiction. □

Having f choose $C = \{c_\alpha : \alpha < \omega_1\} \subseteq \mathbf{R}$ such that $c_\beta \in f(c_\alpha) \to \beta \leqq \alpha$. Let us define $f_1\colon \omega_1 \to [\omega_1]^{\leqq \aleph_0}$ by $\beta \in f_1(\alpha)$ iff $c_\beta \in f(c_\alpha)$. Then by Lemma 2.6, f_1 satisfies W1, and W2 as well. So, applying 2.2 and 2.3 we obtain

Corollary 2.7. *Let* $f\colon \mathbf{R}\to \mathrm{NWD}$ *and* $A_n \in \mathbf{K}^+$ *for* $n<\omega$.

(1) *There is* $A\subseteq \mathbf{R}$ *free for* f, *such that* $|A\cap A_n|\geqq \aleph_0$ *for every* n.

(2) *Assuming* MA + $\neg$CH, *there is* $A\subseteq \mathbf{R}$ *free for* f, *such that*

$$|A\cap A_n| = \aleph_1 \qquad \text{for every } n. \qquad \square$$

Remarks. A typical collection of $\langle A_n \colon n<\omega\rangle$ is the collection of all rational intervals. Thus one can deduce the existence of dense free sets. (1) was, in fact, already proved by F. Bagemihl [2]. In [1] U. Avraham essentially proved, assuming MA + $\neg$CH, the existence of an uncountable free set for $f\colon \mathbf{R}\to \mathrm{NWD}$.

The following question concerning 2.4(2) may arise: What can we say about the existence of free sets for $f\colon \mathbf{R}\to \mathrm{NWD}$ of other cardinalities under MA assumption? This is problem 2 from [8]. We give the full answer to it in the next section. Notice that 2.7(1) answers problems 3 and 4 from [8].

3. On nowhere dense set mappings under MA assumption

In this section we prove the main result of this paper. This is:

Theorem 3.1 (MA). *Assume that* $f\colon \mathbf{R}\to \mathrm{NWD}$, $\kappa<2^{\aleph_0}$ *and* $A_n\in \mathbf{K}^+$ *for* $n<\omega$. *Then there is* $A\subseteq \mathbf{R}$ *free for* f *such that* $|A\cap A_n| = \kappa$ *for every* $n<\omega$.

In [8], S. Hechler noticed that MA implies that there is $f\colon \mathbf{R}\to \mathrm{NWD}$ without free set of power continuum. Thus 3.1 gives us the full description of the possible cardinalities of free sets for $f\colon \mathbf{R}\to \mathrm{NWD}$ (assuming MA). For regular $\kappa<2^{\aleph_0}$ there is an easy argument following that of Avraham ([1], [6]) which yields a free set of cardinality κ for f (assuming MA). Namely, let $\{a_i \colon i<\kappa\}$ be such that if $i<j$ then $a_j\notin f(a_i)$. Let $P=\{\bar{a}\colon \bar{a}=\{a_{i_0},\dots,a_{i_n}\}$ is free for $f\}$ ordered by reverse inclusion. Then P is ccc and there is a $p\in P$ such that for all $q\leqq p$ and $\alpha<\kappa$, there is a $q^1\leqq q$ and a $j>\alpha$ such that $a_j\in q^1$. Thus, MA implies the existence of a free set for f of power κ. Of course the point here is in proving that P is ccc. This can be seen by Lemma 2.6. We do not give here a more detailed proof as 3.1 extends this result. This shows, however, that the main point in the proof of 3.1 is to deal with the case $2^{\aleph_0}=\kappa^+$ for κ singular and to get κ-dense free sets.

Proof of 3.1. W.l.o.g. $f(x)$ is closed and $x\in f(x)$ for all x. We say that a set $S\subseteq \mathbf{R}$ is f-nowhere dense if $f(S)\in \mathrm{NWD}$. From 2.7(1) it follows that it is sufficient to prove 3.1 under $\neg$CH assumption. Let $\kappa<2^{\aleph_0}$. By 2.7(1) we may assume that $\kappa>\aleph_0$.

LEMMA 3.2. *Assume that* $\mathrm{cf}(\kappa)>\aleph_0$, $A\in\mathbf{K}^+$ *and* D *is a family of closed nowhere dense subsets of* $\mathbf{R}$, $|D|<2^{\aleph_0}$. *Then there is* $B\subseteq A$, $|B|=\kappa$ *such that* B *is free for* f, f*-nowhere dense and* $\bigcup D\cap \mathrm{cl}(B)=\emptyset$.

PROOF OF 3.2 is divided into two cases. We can assume that $A\cap\bigcup D=\emptyset$. Let $\{I_n: n<\omega\}$ be an enumeration of all rational intervals of $\mathbf{R}$. Choose $c=\{c_\alpha:\alpha<\kappa\}\subset A$ s.t. $c_\beta\in f(c_\alpha)\rightarrow\beta\leqq\alpha$.

Case 1. κ is regular. Let us consider the following notion of forcing:

$P_0(C)=\{(S,h,g)$: $S\in[C]^{<\aleph_0}$, $g,h\in{}^{\omega>}\omega$ and for κ-many $c\in C$ the following holds
(1) $S\cup\{c\}$ is free for f,
(2) $S\cup\{c\}\subseteq\bigcup\{I_{g(n)}: n\in\mathrm{dom}\, g\}$,
(3) $f(S\cup\{c\})\cap\bigcup\{I_{h(n)}: n\in\mathrm{dom}\, h\}=\emptyset\}$

ordered by: $(S_1,h_1,g_1)\geqq(S_2,h_2,g_2)$ iff $S_1\subseteq S_2$, $h_1\subseteq h_2$ and $\bigcup\{I_{g_2(n)}: n\in\mathrm{dom}\, g_2\}\subseteq\bigcup\{I_{g_1(n)}: n\in\mathrm{dom}\, g_1\}$.

Clearly it suffices to prove the following

CLAIM 3.3. (1) $P_0(C)$ satisfies ccc.

(2) For every $\alpha<\kappa$ the set $\{(S,h,g)\in P_0(C)$: $S\not\subseteq\{c_\beta:\beta<\alpha\}\}$ is dense.

(3) For every $n<\omega$ the set $\{(S,h,g)\in P_0(C)$: $\exists m\in\mathrm{dom}\, h$, $I_{h(m)}\subseteq I_n\}$ is dense.

(4) For every closed $E\in\mathrm{NWD}$, if $E\cap C=\emptyset$ then

$$\{(S,h,g)\in P_0(C): \mathrm{cl}(\bigcup\{I_{g(n)}: n\in\mathrm{dom}\, g\})\cap E=\emptyset\}\text{ is dense.}$$

REMARK. 3.3 is true for singular κ of uncountable cofinality as well, but for further considerations it is sufficient to prove it for regular κ only.

PROOF OF 3.3. (1) Let $\{(S_\alpha,h_\alpha,g_\alpha):\alpha<\omega_1\}\subseteq P_0(C)$. W.l.o.g. (by the Δ-system lemma) for some S, h, g we have $h_\alpha=h_\beta=h$, $g_\alpha=g_\beta=g$ and $S_\alpha\cap S_\beta=S$ for any $\alpha\neq\beta$. Let $S'_\alpha=S_\alpha\setminus S$. We may assume that $|S'_\alpha|=|S'_\beta|=n+1$ for any $\alpha\neq\beta$. For $\alpha<\omega_1$ we define

$$C_\alpha=\{c\in C: S_\alpha\cup\{c\}\text{ is free for } f,\ c\in\bigcup\{I_{g(n)}: n\in\mathrm{dom}\, g\}\ \&\ f(c)\cap\bigcup\{I_{h(n)}: n\in\mathrm{dom}\, h\}=\emptyset\}.$$

For any $c\in C_\alpha$ we choose $j_c,k_c\in{}^{\omega>}\omega$ such that

$$S_\alpha\subseteq\bigcup\{I_{j_c(n)}: n\in\mathrm{dom}\, j_c\}\subseteq\mathbf{R}\setminus f(c)\ \text{ and }\ c\in\bigcup\{I_{k_c(n)}: n\in\mathrm{dom}\, k_c\}\subseteq\mathbf{R}\setminus f(S_\alpha).$$

Choose $j_\alpha,k_\alpha\in{}^{\omega>}\omega$ s.t. $|\{c\in C_\alpha: j_c=j_\alpha\ \&\ k_c=k_\alpha\}|=\kappa$, and then we may assume that $j_\alpha = j_\beta$ and $k_\alpha = k_\beta$ for $\alpha < \beta < \omega_1$. Now for $\alpha, \beta < \omega_1$, for κ-many $c \in C$, $S_\alpha \cup S_\beta$ and c satisfy conditions (2), (3) from the definition of $P_0(C)$ (with h, g common for all α), $f(c)$ is disjoint to $S_\alpha \cup S_\beta$ and $c \not\in f(S_\alpha \cup S_\beta)$. So it is sufficient to prove that there are $\alpha < \beta < \omega_1$ s.t. $S_\alpha \cup S_\beta$ is free for f (then (S_α, h, g), (S_β, h, g) are compatible). Let $S'_\alpha = \{s_0^\alpha, \dots, s_n^\alpha\}$. W.l.o.g. we may assume that for any $k \leqq n$, the sequence $\langle s_k^\alpha : \alpha < \omega_1 \rangle$ is strictly increasing w.r.t. the order induced on C by the order on κ. For $\alpha < \omega_1$ we find $\langle k_0^\alpha, \dots, k_n^\alpha \rangle$ s.t.

$$(*) \qquad \forall i \leqq n \ \forall j \neq i, \quad s_i^\alpha \in I_{k_i^\alpha} \subseteq \mathbf{R} \setminus f(s_j^\alpha).$$

W.l.o.g. for all $\alpha < \beta < \omega_1$ we have $\langle k_0^\alpha, \dots, k_n^\alpha \rangle = \langle k_0^\beta, \dots, k_n^\beta \rangle$, hence

$$\forall \alpha \qquad |\{\beta : f(S_\alpha) \cap S_\beta \neq S\}| \leqq \aleph_0,$$

as $f(S'_\alpha) \cap S'_\beta = \{s_i^\beta : i \leqq n \ \& \ s_i^\beta \in f(s_i^\alpha)\}$ (it follows from (*)). Now define $f_1 : \omega_1 \to [\omega_1]^{\leqq \aleph_0}$ by $\alpha \in f_1(\beta)$ iff $S'_\alpha \in f^*(S'_\beta)$. By Lemma 2.6, f_1 satisfies W1 (and even W2), hence there are $\alpha < \beta < \omega_1$ such that $S_\alpha \cup S_\beta$ is free for f.

(2) Let $(S, h, g) \in P_0(C)$ and $\alpha < \kappa$. Consider $E = \{c_\beta : \beta > \alpha, S \cup \{c_\beta\}$ is free for f, $c_\beta \in \bigcup\{I_{g(n)} : n \in \operatorname{dom} g\}$ & $f(c_\beta) \cap \bigcup\{I_{h(n)} : n \in \operatorname{dom} h\} = \emptyset\}$. $|E| = \kappa$. $f \upharpoonright E$ satisfies the condition analogous to W1 (by Lemma 2.6), hence as in the proof of 2.2 we get that the set

$$B = D \cap \bigcup_{\beta < \kappa} \bigcap \{f(c_\gamma) : c_\gamma \in E \ \& \ \gamma > \beta\}$$

has cardinality $< \kappa$. So there is $c_\beta \in D \setminus B$. Now

$$|\{c \in E : S \cup \{c_\beta, c\} \text{ is free for } f\}| = \kappa,$$

that finishes the proof of (2).

(3), (4) Simple, looking at the proof of (1).

Case 2. κ is singular and $\operatorname{cf}(\kappa) > \aleph_0$. Let $\mu = \operatorname{cf}(\kappa)$ and let $\{\kappa_\alpha\}_{\alpha < \mu}$ be a cofinal in κ sequence of uncountable, regular cardinals. We define by induction on $\alpha < \mu$ a family $S = \{s_\alpha : \alpha < \mu\}$ s.t.

(1) $s_\alpha \subseteq A$, $|s_\alpha| = \kappa_\alpha$ for $\alpha < \mu$,

(2) s_α is bounded, free for f and f-nowhere dense for $\alpha < \mu$,

(3) $\operatorname{cl}(s_\beta) \cap \operatorname{cl}(f(s_\alpha)) = \emptyset$ for $\alpha < \beta < \mu$.

Suppose that we have defined s_β for $\beta < \alpha$. We can find s_α as required in virtue of Lemma 3.2 applied to A and $D = \{\operatorname{cl}(f(s_\beta)) : \beta < \alpha\}$, as for regular κ 3.2 is already proved. Now, having constructed S we can define the following notion of forcing:

$P_1(S) = \{(s, h, g) \colon s \in [\mu]^{<\aleph_0},\ h, g \in {}^{\omega>}\omega$ and for μ-many $\alpha < \mu$ the following holds:
(1) $s \cup \{\alpha\}$ is free for f',
(2) $\bigcup\{\mathrm{cl}(s_\beta) \colon \beta \in s \cup \{\alpha\}\} \subseteq \bigcup\{I_{g(n)} \colon n \in \mathrm{dom}\, g\}$,
(3) $\bigcup\{f(s_\beta) \colon \beta \in s \cup \{\alpha\}\} \cap \bigcup\{I_{h(n)} \colon n \in \mathrm{dom}\, h\} = \emptyset\}$,

ordered by: $(s, h, g) \geqq (s_1, h_1, g_1)$ iff $s \subseteq s_1$, $h \subseteq h_1$ and

$$\bigcup\{I_{g_1(n)} \colon n \in \mathrm{dom}\, g_1\} \subseteq \bigcup\{I_{g(n)} \colon n \in \mathrm{dom}\, g\},$$

where $f' \colon \mu \to \mathcal{P}(\mu)$ is defined by $\beta \in f'(\alpha) \leftrightarrow \mathrm{cl}(s_\beta) \cap \mathrm{cl}(f(s_\alpha)) \neq \emptyset$.

The definition of $P_1(S)$ is similar to that of $P_0(C)$. Thus the proof of the following claim is analogous to that of 3.3. In the proof to separate $\mathrm{cl}(s_\alpha)$ from $\mathrm{cl}(f(s_\beta))$ we use a finite sum of rational intervals instead of a single one, but there are countably many such sums, so the proof can be carried out.

Claim 3.4. (1) $P_1(S)$ satisfies ccc.

(2) For every $\alpha < \mu$ the set $\{(s, h, g) \in P_1(S) \colon s \nsubseteq \alpha\}$ is dense.

(3) For every $n < \omega$ the set $\{(s, h, g) \in P_1(S) \colon \exists m \in \mathrm{dom}\, h,\ I_{h(m)} \subseteq I_n\}$ is dense.

(4) For every closed $E \in \mathrm{NWD}$, if $E \cap \bigcup S = \emptyset$ then

$$\{(s, h, g) \in P_1(S) \colon \mathrm{cl}(\bigcup\{I_{g(n)} \colon n \in \mathrm{dom}\, g\}) \cap E = \emptyset\} \quad \text{is dense.} \qquad \square$$

From 3.4 we easily infer 3.2. □

Now we can prove Theorem 3.1. Let $\{\kappa_n\}_{n<\omega}$ be a sequence of cardinals of uncountable cofinality such that $\kappa = \Sigma_{n<\omega}\kappa_n$. We choose a family $S = \{s_\alpha \colon \alpha < \omega_1\}$ such that

(1) s_α is free for f, f-nowhere dense and bounded for every $\alpha < \omega_1$,

(2) $\mathrm{cl}(s_\beta) \cap \mathrm{cl}(f(s_\alpha)) = \emptyset$ for every $\alpha < \beta < \omega_1$,

(3) for every $n, m < \omega$ the set $B_{n,m} = \{\alpha \colon s_\alpha \subseteq A_n\ \&\ |s_\alpha| = \kappa_m\}$ is uncountable.

This definition is possible by Lemma 3.2. Now let us define $f' \colon \omega_1 \to [\omega_1]^{\leqq\aleph_0}$ by

$$\beta \in f'(\alpha) \leftrightarrow \mathrm{cl}(s_\beta) \cap \mathrm{cl}(f(s_\alpha)) \neq \emptyset.$$

Using Lemma 2.6 one can prove that f' satisfies W1 (and even W2), hence 3.1 follows from 2.2. □

Notice that from the proof of 3.1 it follows that if $f \colon \mathbf{R} \to \mathrm{NWD}$ then there is an infinite $B \subseteq \mathbf{R}$ free for f and f-nowhere dense (without MA assumption).

Topological assumptions (of being nowhere dense) are not the only ones ensuring the existence of large free sets for f. For certain other set mappings with uncountable values we can also prove, assuming MA, the existence of large free sets. Below we show an example of such an argument for f satisfying the condition from Example 1 in Section 2.

For $f\colon \kappa \to \mathcal{P}(\kappa)$ let $P(f)=\{C\in[\kappa]^{<\aleph_0}\colon C$ is free for $f\}$ ordered by reverse inclusion. We can prove

LEMMA 3.5. *Assume that $\kappa>\aleph_0$, $\gamma<\omega_1$ and $f\colon \kappa\to\mathcal{P}(\kappa)$ satisfies*

$$\mathrm{ot}(f(x)\cap f(y))<\gamma \qquad \textit{for any } x\neq y\in\kappa.$$

Then $P(f)$ satisfies ccc.

PROOF OF 4.5. Assume that $\{\bar{a}_\alpha\colon \alpha<\omega_1\}\subseteq P(f)$, $\bar{a}_\alpha\cap\bar{a}_\beta=\varnothing$ for $\alpha<\beta<\omega_1$ and $|\bar{a}_\alpha|=n+1$ for every α. We shall prove

$$(*)\qquad \begin{array}{c}\text{There is an uncountable } C\subseteq\omega_1 \text{ such that for every } \alpha\in C\\ \text{the set } \{\beta\in C\colon \bar{a}_\alpha\cap f(\bar{a}_\beta)\neq\varnothing\} \text{ is countable.}\end{array}$$

To see it, let $\bar{a}_\alpha=\{a^0_\alpha,\dots,a^n_\alpha\}$. For $i,j\leqq n$ let $C_{ij}(\alpha)=\{\beta\in\omega_1\colon a^i_\alpha\in f(a^j_\beta)\}$. Now for $\alpha<\omega_1$, $\{\beta\in\omega_1\colon \bar{a}_\alpha\cap f(\bar{a}_\beta)\neq \beta\}=\bigcup_{i,j\leqq n}C_{ij}(\alpha)$, hence it suffices to choose uncountable sets $C_{0,0}\supseteq C_{0,1}\supseteq,\dots,\supseteq C_{n,n}$ such that for every $\alpha\in C_{ij}$ we have $C_{ij}(\alpha)\cap C_{ij}$ is countable. Then $C=C_{nn}$ will satisfy $(*)$. First we have to choose $C_{0,0}$. If for every $\alpha<\omega_1$, $C_{0,0}(\alpha)$ is countable, then we take $C_{0,0}=\omega_1$. Otherwise there is $\alpha<\omega_1$ such that $C_{0,0}(\alpha)$ is uncountable. Then by hypotheses of 3.5 we can take $C_{0,0}=C_{0,0}(\alpha)\setminus\{\alpha\}$. Similarly we define other C_{ij}'s, thus proving $(*)$.

Now having found $C\subseteq\omega_1$, notice that $f'\colon C\to[C]^{\leqq\aleph_0}$ defined by $\alpha\in f'(\beta)$ iff $\bar{a}_\alpha\cap f(\bar{a}_\beta)\neq\varnothing$ satisfies W1. So by 2.2 we get $\alpha\neq\beta$ such that $\bar{a}_\alpha\cup\bar{a}_\beta\in P(f)$. □

REMARK. Notice that 3.5 enables us to drop in 2.4 the assumption that $f(x)$ is countable.

4. Further results on nowhere dense set mappings

In this section we complete an answer to problems from [8]. In adjusted notation, problem 1 from [8] reads: Does the existence of a nowhere dense set mapping with values of power $<2^{\aleph_0}$ without uncountable free set imply CH? U. Avraham proved in [1] that it is consistent with ZFC $+\neg$CH that there is $f\colon \mathbf{R}\to\mathrm{NWD}$ without uncountable free set, thus partially answering this problem. In the following lemma we give another proof of this fact.

LEMMA 4.1. *Let M be a countable transitive model of* ZFC. *Then in $\mathcal{R}_{\omega_1}$ generic extension of M there is $f: \mathbf{R} \to \mathrm{NWD}$ without uncountable free set.*

PROOF. Let $N = M[\langle r_\alpha : \alpha < \omega_1 \rangle]$, where $\langle r_\alpha : \alpha < \omega_1 \rangle$ is a sequence of random reals. Let $A \dot{\cup} B = \mathbf{R}$ be a Borel partition of $\mathbf{R}$ such that $A \in \mathbf{K}$ and the Lebesgue measure of B equals zero. $A = \bigcup \{\# f_n : n < \omega\}$ where $f_n \in {}^\omega\omega$ and $\# f_n$ is closed and nowhere dense. Here $\# f$ denotes the Borel set coded by f. It is well known that if r is random over M then $M \cap \mathbf{R} \subseteq A + r = \bigcup \{\# f_n + r : n < \omega\}$.

CLAIM 4.2. Assume that $\bar{a} \in {}^\omega\mathbf{R}$ is a sequence converging to $b \in \mathbf{R}$, $\bar{a}, b \in M$, r is random over M and moreover $b \in \# f_n + r$. Then for some $n < \omega$, $\bar{a}(m) \in \# f_n + r$.

PROOF OF 4.2. Notice that the set $(b - \# f_n) \setminus \bigcup_{m<\omega} (\bar{a}(m) - \# f_n)$ has the Lebesgue measure zero. □

Now we can construct $f: \mathbf{R} \to \mathrm{NWD}$ in N without uncountable free set. Let $x \in \mathbf{R}$. Take the first $\alpha < \omega_1$ such that $x \in M[\langle r_\beta : \beta < \alpha \rangle]$ and the first $n < \omega$ such that $x \in \# f_n + r_\alpha$ and define

$$f(x) = (\# f_n + r) \cap M[\langle r_\beta : \beta < \alpha \rangle].$$

Assume now that C is uncountable and free for f. W.l.o.g. $C = \{c_\alpha : \alpha < \omega_1\}$ and for every $\alpha < \omega_1$, $\{c_\beta : \beta < \alpha\} \in M[\langle r_\beta : \beta < \gamma \rangle]$ where γ is the first ordinal such that $c_\alpha \in M[\langle r_\beta : \beta < \gamma \rangle]$. From 4.2 it follows that for every $\alpha < \omega_1$, $c_\alpha \notin \mathrm{cl}\{c_\beta : \beta < \alpha\}$, a contradiction. □

To complete the answer to problem 1 from [8] it is sufficient to construct a generic model of ZFC + $\neg$CH in which there is $f: \mathbf{R} \to \mathrm{NWD}$ without uncountable free set, such that $|f(x)| < 2^{\aleph_0}$. The simplest way to do it is by adding $\aleph_{\omega_1}$ random reals to a model of ZFC + CH. The proof is the same as in 4.1.

We shall give yet an answer to problem 5 from [8], which reads: Assume that $f: \mathbf{R} \to \mathrm{NWD}$. Are there $A, B \subseteq \mathbf{R}$, $|A| = |B| = 2^{\aleph_0}$ such that for every $a \in A$, $b \in B$, $\{a, b\}$ is free? The positive answer is given in 4.3.

Let $\mathbf{K}^* = \{A \in \mathbf{K}^+ : \forall n,\ I_n \cap A \neq \emptyset \to \forall B \in \mathbf{K} \,|(A \cap I_n) \setminus B| = 2^{\aleph_0}\}$, where $\{I_n : n < \omega\}$ is an enumeration of all rational intervals of $\mathbf{R}$.

FACT 4.3. Let $f: \mathbf{R} \to \mathrm{NWD}$ and $A \in \mathbf{K}^*$. Then there are disjoint nonempty $B, C \in \mathbf{K}^*$, $B, C \subseteq A$, such that $\{x, y\}$ is free for f for any $x \in B$, $y \in C$.

PROOF. Similar to that in Maté [9]. First, let $T = \{n < \omega : I_n \cap A \neq \emptyset\}$. For $n \in T$ let $A_n = \{x \in A : f(x) \cap I_n = \emptyset\}$. We have $A = \bigcup \{A_n : n \in T\}$, and one can prove that for some n and m, $A_n \cap I_m \in \mathbf{K}^*$. W.l.o.g. $I_n \cap I_m = \emptyset$. Let $B^0 = A_n \cap I_m$ and $C^0 = A \cap I_n$. Now, $y \notin f(x)$ for any $x \in B^0$, $y \in C^0$. Repeating the above operation once more, we get the conclusion of 4.3. □

Thus we have presented answers to all problems from [8]. Let us finish with the following questions:

QUESTION 4.4. (MA + ¬CH). Let $f: \mathbf{R} \to \mathrm{NWD}$ and $A_\alpha \in \mathbf{K}^+$ for $\alpha < \omega_1$. Does there exist $B \subseteq \mathbf{R}$ free for f, such that $B \cap A_\alpha \neq \emptyset$ for every $\alpha < \omega_1$?

QUESTION 4.5. Con(ZFC + $\mathrm{cov}\,\mathbf{K} > \omega_1$ + ($\exists f: \mathbf{R} \to \mathrm{NWD}$) ($f$ has no uncountable free set))?

QUESTION 4.6 (MA). Let $f: \mathbf{R} \to \mathrm{NWD}$ and $\kappa < 2^{\aleph_0}$, $\mathrm{cf}(\kappa) = \aleph_0$. Does there exist $B \subseteq \mathbf{R}$ free for f, f-nowhere dense and of cardinality κ?

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